Lesson 3: Understanding the First and Second Derivatives

The rate of change of the rate of change of a function can be calculated using the **second derivative**. This concept is used to determine whether the rate of change is increasing or decreasing at a certain point.

Given that ,

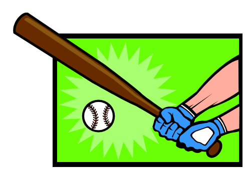
then, , (using the power and the sum or difference rules)

and it follows that .

One of the most commonly used applications of higher-order derivatives is the concept of position or **displacement**, **velocity** and **acceleration.** The **velocity** of a car is defined as the rate of change of displacement at a specific point in time, while the **acceleration** is the rate of change of the velocity of the car at a specific point in time.

If the position of an object is represented by ***s(t)***, then the object's velocity ***v(t) = s'(t).*** Or, in other words, the velocity is equal to the derivative of the position. It follows that the acceleration of the object ***a(t) = v'(t) = s''(t)***, or that the acceleration is the derivative of the velocity, or the second derivative of the position.

Example #2:



A baseball is hit vertically upward. The position function $s(t)$, in metres, of the ball above ground is $s(t) = -5t^2 + 30t + 1$, where t is the time in seconds.

a) Determine the maximum height reached by the ball.

When the ball reaches the maximum height, it stops instantaneously before changing direction. So, to determine the maximum height, find when the velocity of the ball is zero – so find s'(t), then set equal to zero!

$s(t) = -5t^2 + 30t + 1$

$s'(t) = -10t + 30 = 0$
$30 = 10t$
$3 = t$ The ball reaches the maximum height at 3 s.

b) Determine the velocity of the ball when it is caught 1m above the ground.

To determine velocity, $v(t) = s'(t)$... but you need a value for TIME, not just a height of 1m. First, find the TIME at which the height of the ball is at 1m.

http://www.pineryparkll.com/Portals/0/Baseball%20in%20flight.jpg

$s(t) = -5t^2 + 30t + 1 = 1$ → solve the quadratic only if it equals zero
$-5t^2 + 30t = 0$ → now factor to solve
$-5t(t - 6) = 0$
$t = 0 \text{ or } 6$ So, the ball reaches a height of 1m after 6 seconds. (since 0 is inadmissible).

Now use the value t = 6s in the equation for velocity:

v(t) = s'(t) = -10t + 30

v(6) = -10(6) + 30

= -60 + 30

= -30 (This negative value means the ball is travelling downwards.)

The velocity of a ball that is caught 1m above the ground is -30m/s OR 30m/s [down].

c) Determine the acceleration of the ball at 1 second.

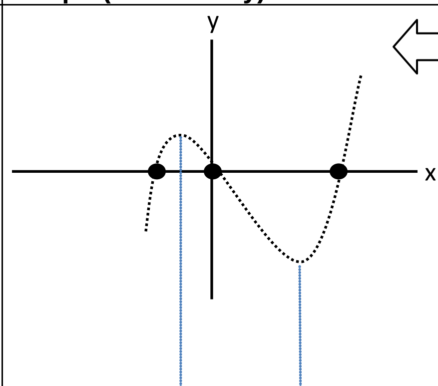
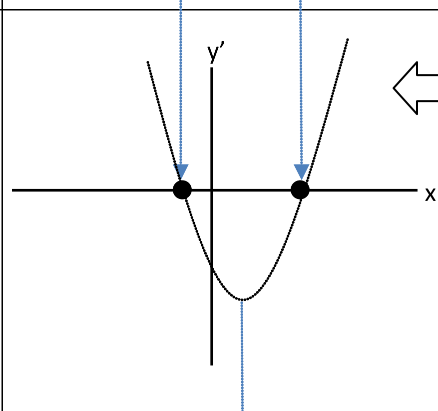
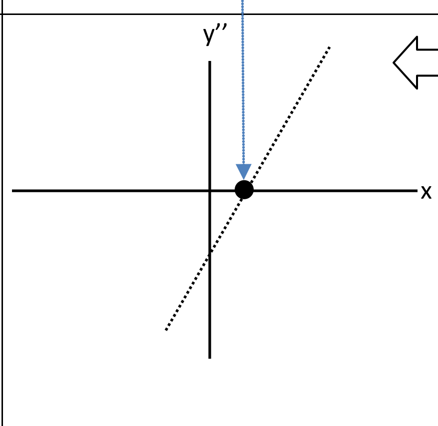
To determine acceleration, a*(t) = v'(t) = s''(t)*.

v(t) = -10t + 30 *a(t) = v'(t) = -10*

The acceleration of the ball at 1 second (or at **any** time) is -10 m/s/s or 10 m/s^2 [down].

<u>Example #3:</u>

Complete the chart by determining both the equation of the first and second derivatives, then also sketching graphs of the function, the first and second derivatives:

	Equation	**Graph (sketch only)**
Original Function	$f(x) = x^3 – 2x^2 – 3x$ (factor to find the zeros) $= x(x^2 – 2x – 3)$ $= x(x – 3)(x + 1)$ Zeros of this function exist at x = -1, 0, 3 *(This cubic function has a positive leading coefficient, so will start low and end high.)*	
First Derivative	$f'(x) = 3x^2 – 4x – 3$ (use quad formula to find zeros) This function has two zeros at approximately x = -0.53, 1.87 *(This quadratic function has a positive leading coefficient, so the graph will start high and end low, as it travels through both zeros.)*	
Second Derivative	f''(x) = 6x - 4 (factor to find the zeros) = 2(3x – 2) This function has one zero at x = – *(This linear function has a positive leading coefficient, so the graph will be a line with a positive slope –going uphill.)*	

This graph goes uphill, hits a maximum point, then goes downhill, hits a minimum point, then goes uphill once more.

The value of the derivative (or rate of change) at the max and min points is zero, so the graph of the first derivative must have zeros that are "in-line" with the max and min points of the original function (see blue arrows).
The values of the derivative are positive when the original curve is increasing, and negative when the original curve is decreasing.

Thus, the curve of f'(x) will be above the x-axis when f(x) is increasing, and below the x-axis when the original function is decreasing, then be above the x-axis once more when the original function is increasing.

The graph of f'(x) has one minimum value, so the graph of f''(x) will have one zero that corresponds with this x-value (blue arrow).

The graph for the f''(x) function must be below the x-axis (be negative) when the graph of f'(x) is decreasing, then be above the x-axis when the graph of the first derivative is increasing once again.

(For additional instruction or clarification, you may choose to refer to an on-line tutorial on Graphing Derivative Functions at <u>http://www.youtube.com/watch?v=Gbtma_UQpro&feature=related</u>)

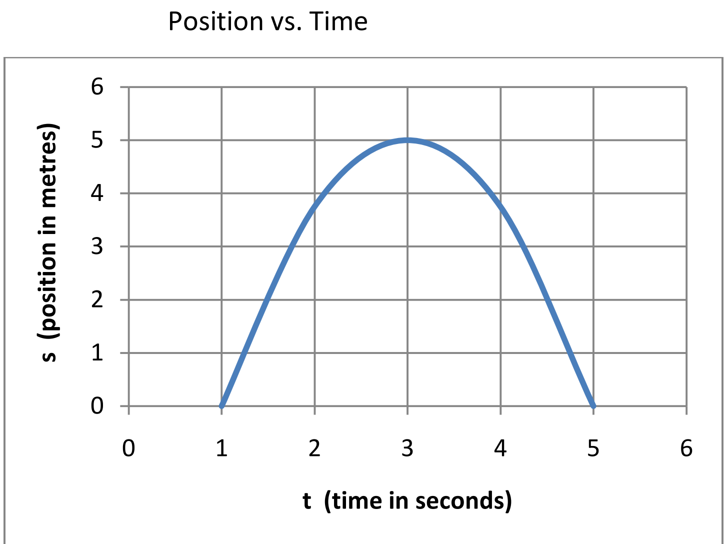
Provide complete solutions to **Lesson 3 questions**: *(complete solutions found in Appendix A)*
(answer key found in Appendix B)

1. Each of the following position functions describes the motion of an object along a straight line. Find the velocity AND acceleration as functions of time, t ≥ 0:

   a) b) –

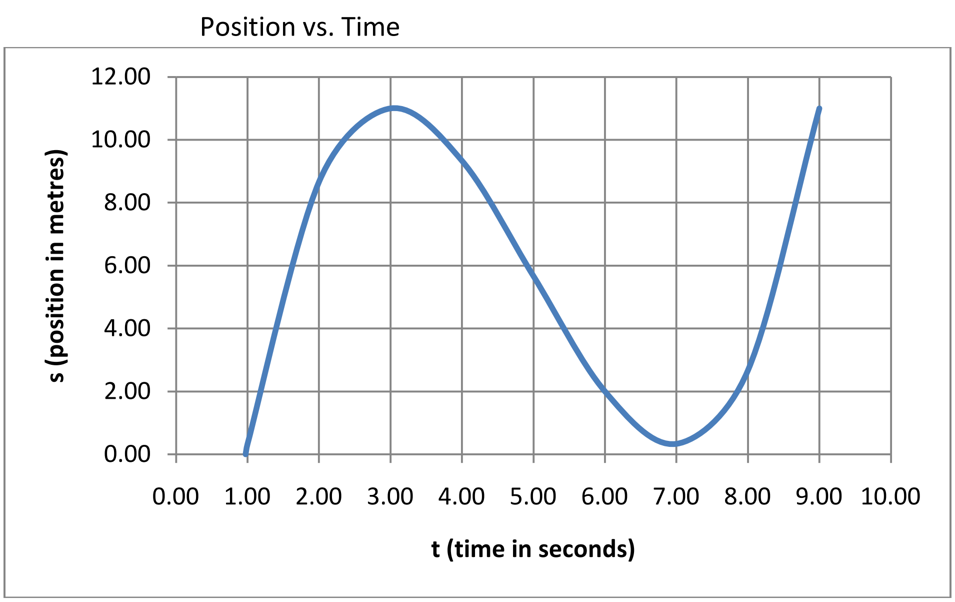
2. Refer to each position versus time graphs below when answering the following:

a)



i) When is the velocity zero?
ii) When is the object moving in a positive direction?
iii) When is the object moving in a negative direction?

b)



i) When is the velocity zero?
ii) When is the object moving in a positive direction?
iii) When is the object moving in a negative direction?

3. A ball is thrown upward, and its height, h, in metres above the ground after t seconds is given by the equation

a) Determine expressions for the velocity and acceleration of the ball.
b) Calculate the ball's initial velocity.
c) Calculate the maximum height of the ball?
d) When does the ball hit the ground?
e) What is the ball's velocity at the instant it hits the ground?